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LC SLANT HELIX ON HYPERSURFACES IN MINKOWSKI SPACE $\mathbb{E}_1^{n+1}$

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Abstract. In this paper we give the definition of a LC slant helix for a non-null curve lying on a hypersurface in $\mathbb{E}_1^{n+1}$ by using the Levi Civita's notion of parallel vector field. Also we define a vector field D which we called Darboux vector field of LC slant helix on a hypersurface in $\mathbb{E}_1^{n+1}$. Morever we give some basic properties and characterization of LC slant helices.

Keywords: LC slant helix, hypersurface, Minkowski space.

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1. Introduction

A curve of constant slope or general helix in Euclidean $3-$space $\mathbb{E}^3$, is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 ([15]) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. If both of k_1 and k_2 are non-zero constants it is, of course, a general helix. It is known that straight line and circle are degenerate helix examples. ($k_1 = 0$, if the curve is straight line and $k_2 = 0$, if the curve is a circle)([8]).

The notion of a generalized helix can be generalized to higher dimensions in many ways. In[13] the same definition is proposed but in $\mathbb{E}^n$. In [5] the definition which was defined by Hayden is more restrictive: the fixed direction makes a constant angle with all the vectors of the Frenet frame. This definition only works in the odd dimensional case. Moreover, in the same reference, it is proved that the definition is equivalent to the fact that the ratios $\frac{k_{n-1}}{k_{n-2}}, \frac{k_{n-3}}{k_{n-4}}, ..., \frac{k_2}{k_1}$ being the curvatures, are constant. This statement is related with the Lancret Theorem for generalized helices in $\mathbb{E}^3$ (the ratio of torsion to curvature is constant). In [10] the curves in $\mathbb{E}^n$ for which all the ratios $\frac{k_{n-1}}{k_{n-2}}, \frac{k_{n-3}}{k_{n-4}}, ..., \frac{k_2}{k_1}$ are constant which was called *ccr* curves. In the same reference, it is shown that in the even dimensional case, a curve has constant curvature ratios if and only if its tangent indicatrix is a geodesic in the flat torus.

Izumiya and Takeuchi defined a new kind of helix which is called slant helix([7]) and they gave a characterization of slant helices in Euclidean $3-$space $\mathbb{E}^3$.Then Kula and Yaylı investigated spherical images the tangent indicatrix and binormal indicatrix of a slant helix ([9]). Morever, they gave a characterization for slant helices in $\mathbb{E}^3$: "For involute of a curve γ, γ is a slant helix if and only if its involute is a general helix". In 2008, Önder *et al.* defined a new kind of slant helix in Euclidean $4-$space $\mathbb{E}^4$ which they called B_2-slant helix and they gave some characterizations of this slant helix in Euclidean $4-$space $\mathbb{E}^4$([11]).

In this paper, we give the definition of a LC slant helix for a non-null curve lying on a hypersurface in $\mathbb{E}_1^{n+1}$ by using the Levi Civita's notion of parallel vector field, as follows:

$$\langle V_n, X \rangle = \text{constant},$$

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where V_n is $n-$th Frenet vector field of the curve and X is Levi Civita's notion of parallel vector field. Also we give a definition of harmonic curvatures functions in terms of V_n, and we define a new vector field D which we called Darboux vector field of LC slant helix on a hypersurface in $\mathbb{E}_1^{n+1}$ Morever we give some basic properties and characterization of LC slant helices.

2. Preliminaries

Let E_1^{n+1} be the $(n+1)$ dimensional pseudo-Euclidean space with index 1 endowed with the indefinite inner product given by

$$g(x,y) = -x_1y_1 + \sum_{i=2}^{n+1} x_iy_i,$$

where $x = (x_1, x_2, \cdots, x_{n+1})$, $y = (y_1, y_2, \cdots, y_{n+1})$ is the usual coordinate system. Let M be a hypersurface in E_1^{n+1} and p be a point on M and $v \in T_pM$ a tangent vector. Then v is said to be spacelike, timelike or null according to $g(v,v) > 0$, $g(v,v) < 0$, or $g(v,v) = 0$ and $v \neq 0$, respectively. Notice that the vector $v = 0$ is spacelike. The category into which a given tangent vector falls is called its causal character. These definitions can be generalized for curves as follows. A curve α on M is said to be spacelike if all of its velocity vectors $\alpha'$are spacelike, similarly for timelike and null([1]).

Let us recall from [14, 6] the definition of the Frenet frame and curvatures.

Let M be a hypersurface in E_1^{n+1} and $\alpha : I \subset \mathbb{R} \to M$ be non-null curve on M. A non-null curve $\alpha(s)$ is said to be a unit speed curve if $g(\alpha'(s), \alpha'(s)) = \varepsilon_0$, ($\varepsilon_0$ being +1 or -1 according to α is spacelike or timelike respectively). Let $\{V_1, V_2, ..., V_n\}$ be the moving Frenet frame along the unit speed curve α, where V_i $(i = 1, 2, ..., n)$ denote ith Frenet vector fields and k_i be ith curvature functions of the curve $(i = 1, 2, ..., n-1)$. Then the Frenet formulas are given by

$$\begin{aligned} \nabla_{V_1} V_1 &= k_1 V_2, \\ \nabla_{V_1} V_i &= -\varepsilon_{i-2}\varepsilon_{i-1} k_{i-1} V_{i-1} + k_i V_{i+1}, \quad 1 < i < n \\ \nabla_{V_1} V_n &= -\varepsilon_{n-2}\varepsilon_{n-1} k_{n-1} V_{n-1} \end{aligned} \tag{1}$$

where $g(V_i, V_i) = \varepsilon_{i-1}$, and ∇ is the Levi-Civita connection of M.

Let M be a hypersurface in E_1^{n+1} with the Levi-Civita connection ∇ and suppose $\alpha : I \subset \mathbb{R} \to M$ is non-null curve on M . For any tangent vector X at $\alpha(s)$ is said to be a Levi Civita's notion of parallel vector field on M of a direction along a curve if $\nabla_{V_1} X = 0$. Also, the Levi Civita's notion of parallel vector field has constant lenght([4]).

3. LC slant helix on hypersurfaces in Minkowski space

In this section we define LC slant helices on hypersurfaces in Minkowski space and we give some characterizations for LC slant helices on hypersurfaces in the same space.

Definition 3.1. *Let M be a hypersurface in E_1^{n+1} and $\alpha : I \subset \mathbb{R} \to M$ be non-null curve on M. A non-null curve $\alpha(s)$ is said to be a LC slant helix if there exists a Levi Civita's notion of parallel vector field X on M such that $g(V_n, X)$ is a constant function. Any line parallel this direction X is called the axis of the LC slant helix.*

Definition 3.2. *Let M be a hypersurface in E_1^{n+1} and $\alpha : I \subset \mathbb{R} \to M$ be a unit speed on M. Harmonic curvatures of α is defined by*

$$H_i : I \subset \mathbb{R} \to \mathbb{R},$$

$$\begin{aligned} H_0 &= 0, \\ H_1 &= \varepsilon_{n-3}\varepsilon_{n-2}\frac{k_{n-1}}{k_{n-2}}, \\ H_i &= (k_{n-i}H_{i-2} - \nabla_{V_1}H_{i-1})\,\frac{\varepsilon_{n-(i+2)}\varepsilon_{n-(i+1)}}{k_{n-(i+1)}}, \quad 2 \le i \le n-2, \end{aligned} \tag{2}$$

where $k_1, k_2, ..., k_{n-1}$ are curvatures functions of the curve α which are not necessarily constant.

Theorem 3.3. *Let M be a hypersurface in E_1^{n+1} and $\alpha : I \subset \mathbb{R} \to M$ be a unit speed LC slant helix. Let $\{V_1, V_2, ..., V_n\}$, $\{H_1, H_2, ..., H_{n-2}\}$ be denote the Frenet frame and the higher ordered harmonic curvatures of the curve, respectively. Then the following equation is holds*

$$g(V_{n-(i+1)},\ X) = H_i\ g(V_n,\ X), \quad 1 \le i \le n-2, \tag{3}$$

where X is axis of the LC slant helix.

*Proof.*We use mathematical induction on i. Since X is axis of the LC slant helix α, we get

$$X = \lambda_1 V_1 + \lambda_2 V_2 + ... + \lambda_n V_n.$$

From the definition of LC slant helix we have

$$g(V_n, X) = \lambda_n \varepsilon_{n-1} \tag{4}$$

By taking the derivative of (4) and applying the Frenet formulas twice we obtain

$$g(V_{n-1}, X) = 0, \tag{5}$$

$$g(V_{n-2}, X) = H_1\ g(V_n,\ X).$$

respectively. Hence it is shown that (3) is true for $i = 1$. We now assume (3) is true for the first $i-1$. Then we have

$$g(V_{n-i}, X) = H_{i-1}\ g(V_n,\ X). \tag{6}$$

By taking the derivative of (6) and applying the Frenet formulas, we get

$$-\varepsilon_{n-i-2}\varepsilon_{n-i-1}k_{n-i-1}\ g(V_{n-i-1}, X) + k_{n-i}\ g(V_{n-i+1}, X) = \nabla_{V_1}H_{i-1}\ g(V_n, X).$$

By using the our induction hypothesis, $g(V_{n-i+1}, X) = H_{i-2}\ g(V_n,\ X)$, we have

$$(k_{n-i}H_{i-2} - \nabla_{V_1}H_{i-1})\,\frac{\varepsilon_{n-(i+2)}\varepsilon_{n-(i+1)}}{k_{n-(i+1)}}\ g(V_n,\ X) = g(V_{n-(i+1)}, X),$$

it follows that

$$g(V_{n-(i+1)},\ X) = H_i\ g(V_n,\ X).$$

Theorem 3.4. *Let M be a hypersurface in E_1^{n+1} and non-null curve $\alpha : I \subset \mathbb{R} \to M$ be a unit speed LC slant helix with Frenet vector fields $\{V_1, V_2, ..., V_n\}$, and harmonic curvatures $\{H_1, H_2, ..., H_{n-2}\}$. If X is axis of the LC slant helix α on M, then*

$$X = g(V_n, X)\left(\sum_{j=1}^{n-2} H_j V_{n-(j+1)}\varepsilon_{n-(j+2)} + \varepsilon_{n-1}V_n\right).$$

*Proof.*If the axis of the LC slant helix α on M is X, then we can write

$$X = \sum_{i=1}^{n} \lambda_i V_i.$$

By using the Theorem(3.3) we have

$$\begin{aligned}
\lambda_1 &= \varepsilon_0\ H_{n-2} g(V_n,\ X), \\
\lambda_2 &= \varepsilon_1\ H_{n-3} g(V_n,\ X), \\
&\ \vdots \\
\lambda_{n-2} &= \varepsilon_{n-3} H_1\ g(V_n,\ X), \\
\lambda_{n-1} &= 0, \\
\lambda_n &= \varepsilon_{n-1}\ g(V_n,\ X).
\end{aligned} \tag{7}$$

Thus we can easily obtain

$$X = g(V_n, X) \left(\sum_{j=1}^{n-2} H_j V_{n-(j+1)} \varepsilon_{n-(i+2)} + \varepsilon_{n-1} V_n \right).$$

Corollary 3.5. *Let M be a hypersurface in E_1^{n+1} and non-null curve $\alpha : I \subset \mathbb{R} \to M$ be a unit speed LC slant helix with Frenet vector fields $\{V_1, V_2, ..., V_n\}$, and harmonic curvatures $\{H_1, H_2, ..., H_{n-2}\}$. If α LC slant helix, then $\sum_{i=1}^{n-2} \varepsilon_{n-(i+2)} H_i^2 = c$, where c is any constant.*

Proof. Let α be generalized LC slant helix with the arc length parameter s .Since X is a unit Levi Civita's notion of parallel vector field and from Theorem(3.3) we obtain

$$(g(V_n, X))^2 \left(\varepsilon_{n-1} + \sum_{j=1}^{n-2} \varepsilon_{n-(j+2)} H_j^2 \right) = 1. \tag{8}$$

Thus we get

$$\sum_{j=1}^{n-2} \varepsilon_{n-(j+2)} H_j^2 = \frac{1 - \varepsilon_{n-1} \lambda_n^2}{\lambda_n^2}.$$

for some non zero constant λ_n.

Definition 3.6. *Let M be a hypersurface in E_1^{n+1} and non-null curve $\alpha : I \subset \mathbb{R} \to M$ be a unit speed non- degenerate curve with Frenet vector fields $\{V_1, V_2, ..., V_n\}$, and harmonic curvatures $\{H_1, H_2, ..., H_{n-2}\}$. The vector*

$$D = \sum_{j=1}^{n-2} H_j V_{n-(j+1)} \varepsilon_{n-(j+2)} + \varepsilon_{n-1} V_n$$

is called the Darboux vector of the curve α.

Theorem 3.7. *Let M be a hypersurface in E_1^{n+1} and non-null curve $\alpha : I \subset \mathbb{R} \to M$ be a unit speed curve with Frenet vector fields $\{V_1, V_2, ..., V_n\}$, and harmonic curvatures $\{H_1, H_2, ..., H_{n-2}\}$. Then α is a LC slant helix if and only if*

$$D = \sum_{j=1}^{n-2} H_j V_{n-(j+1)} \varepsilon_{n-(j+2)} + \varepsilon_{n-1} V_n$$

is Levi Civita's notion of parallel vector field.

Proof. Suppose that α is LC slant helix on M and X is axis of α. From Corollary(3.5), we get

$$X = g(V_n, X) \left(\sum_{j=1}^{n-2} H_j V_{n-(j+1)} \varepsilon_{n-(j+2)} + \varepsilon_{n-1} V_n \right) \tag{9}$$

By taking the derivative of (9), and using the definition of the Levi Civita's notion of parallel vector field we can easily obtain

$$\nabla_{V_1} D = 0.$$

Thus D is Levi Civita's notion of parallel vector field.

Conversely, since D is Levi Civita's notion of parallel vector field then $\|D\| = \text{constant}$. We consider the normalisation of the Levi Civita's notion of parallel vector field as follows

$$X = \frac{1}{\|D\|} D.$$

Therefore we have

$$g(V_n,\ X) = \frac{1}{\|D\|} = \text{constant}.$$

Thus α is a LC slant helix.

From now, let us consider M is a hypersurface in E_1^4 and $\alpha : I \subset \mathbb{R} \to M$ be a non-null curve with Frenet vector fields $\{V_1, V_2, V_3\}$ and curvatures $\{k_1, k_2\}$.

Theorem 3.8. *Let M be a hypersurface in E_1^4, and α be a regular curve on M. Then α is a LC slant helix if and only if $\frac{k_2}{k_1}$ is constant.*

Proof. Let M be a hypersurface in E_1^4, and α is a generalized LC helix on M. Without loss of generality, assume α has unit speed. By the definition of LC helix, there exists a Levi Civita's notion of parallel vector field X such that

$$g(V_3, X) = \lambda_3 \varepsilon_2 \tag{10}$$

for some non zero constant λ_3. By taking the derivative of (10) and applying the Frenet formulas

$$\begin{aligned} g(\nabla_{V_1} V_3, X) &= 0, \\ g(-\varepsilon_1 \varepsilon_2 k_2 V_2, X) &= 0. \end{aligned}$$

Since $k_2 \neq 0$, then we get

$$g(V_2, X) = 0.$$

Now, X is perpendicular to V_2, so

$$X = aV_1 + bV_3, \tag{11}$$

for some non zero function a, b. Because X is a unit Levi Civita's notion of parallel vector field, $\nabla_{V_1} X = 0$. By taking the derivative of (11), and applying the Frenet formulas we have

$$0 = a'\ V_1 + (ak_1 - \varepsilon_1 \varepsilon_2 k_2 b)\, V_2 + b'\ V_3.$$

Since $\{V_1, V_2, V_3\}$ are linearly independent we have

$$\begin{gathered} a' = 0, \\ ak_1 - \varepsilon_1 \varepsilon_2 k_2 b = 0, \\ b' = 0. \end{gathered}$$

Hence

$$\frac{k_1}{k_2} = \varepsilon_1 \varepsilon_2 \frac{b}{a} = \text{constant}.$$

Now suppose that $\frac{k_1}{k_2}$ is constant. We can choose $\frac{k_1}{k_2} = \varepsilon_1 \varepsilon_2 c$, for some non zero constant c and define

$$X = \frac{1}{c} V_1 + V_3$$

to get

$$\nabla_{V_1} X = \frac{1}{c} k_1 V_2 - \varepsilon_1 \varepsilon_2 k_2 V_2 = 0.$$

Hence, X is a Levi Civita's notion of parallel vector field and clearly $g(X, V_3) = \varepsilon_2$ is constant. Thus α is a generalized LC slant helix.

Corollary 3.9. *Let M be a hypersurface in E_1^4, and α be a non- degenerate curve on M. From the Definition(3.2) and Definition(3.6) we can write*

$$D = \varepsilon_1 \frac{k_2}{k_1} V_1 + \varepsilon_2 V_3. \tag{12}$$

where k_1 and k_2 are curvatures of the curve. By taking the derivative of (12) we have

$$\nabla_{V_1} D = \varepsilon_1 \left(\frac{k_2}{k_1} \right)' V_1. \tag{13}$$

If all curvatures of the curve are constants, i.e., the curve is a $W-$curve, take the derivative of D we get

$$\nabla_{V_1} D = 0.$$

So, from Theorem (3.7) the curve α is LC slant helix.

Corollary 3.10. *Let M be a hypersurface in E_1^5, and α be a non- degenerate curve on M. From the Definition(3.2) and Definition(3.6) we can write*

$$D = -\varepsilon_1 \frac{1}{k_1} \left(\frac{k_3}{k_2} \right)' + \varepsilon_2 \frac{k_3}{k_2} V_2 + \varepsilon_3 V_4.$$

where k_1, k_2 and k_3 are curvatures of the curve. If all curvatures of the curve are constants, i.e., the curve is a $W-$curve, then we get

$$D = \varepsilon_2 \frac{k_3}{k_2} V_2 + \varepsilon_3 V_4.$$

If we take the derivative of D we get

$$\nabla_{V_1} D = -\varepsilon_0 \varepsilon_1 \varepsilon_2 \frac{k_1 k_3}{k_2} V_1$$

Since α is non-degenerate curve, we obtain that $\nabla_{V_1} D \neq 0$ or D is not Levi Civita's notion of parallel vector field. So, from Theorem (3.7) the curve is not LC slant helix.

Corollary 3.11. *Let M be a hypersurface in E_1^5, and α be a non- degenerate curve on M. If α is a LC slant helix then,*

$$\left[\frac{1}{k_1} \left(\frac{k_3}{k_2} \right)' \right]' + \varepsilon_0 \varepsilon_1 k_1 \frac{k_3}{k_2} = 0.$$

Proof. Let α be LC slant helix . From Corollary(3.5) for $n = 4$, we have $\varepsilon_1 H_1^2 + \varepsilon_0 H_2^2$ =constant. By using the Definition(3.2)

$$\varepsilon_1 \left(\frac{k_3}{k_2} \right)^2 + \varepsilon_0 \left[\frac{1}{k_1} \left(\frac{k_3}{k_2} \right)' \right]^2 = \text{constant}. \tag{14}$$

By taking the derivative of Eq.(14) we obtain

$$\left[\frac{1}{k_1} \left(\frac{k_3}{k_2} \right)' \right]' + \varepsilon_0 \varepsilon_1 k_1 \frac{k_3}{k_2} = 0. \tag{15}$$

Theorem 3.12. *Let M be a hypersurface in E_1^{2m}, and α be a non- degenerate curve on M. and $\{H_1, H_2, ..., H_{2m-3}\}$ be the harmonic curvature functions of the curve α. If the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \frac{k_6}{k_5} ... \frac{k_{2m-2}}{k_{2m-3}}$ are constant, then we have for $2 \leq i \leq m$*

$$\begin{aligned} H_{2i-2} &= 0, \\ H_{2i-3} &= \frac{k_{2m-2}}{k_{2m-3}} . \frac{k_{2m-4}}{k_{2m-5}} ... \frac{k_{2m+1-(2i-3)}}{k_{2m+1-(2i-2)}} \varepsilon_{2m-3}\varepsilon_{2m-4}...\varepsilon_{2m-(2i-2)}. \end{aligned} \tag{16}$$

Proof. We apply the induction method for the proof.
The case of $i = 2$:
From the definition of harmonic curvature functions of α we can write

$$H_2 = -\left(\frac{k_{2m-2}}{k_{2m-3}}\right)' \frac{\varepsilon_{2m-3}\varepsilon_{2m-5}}{k_{2m-4}}.$$

By using the hypothesis we have

$$H_2 = 0.$$

Again from Definition(3.2) we have

$$H_1 = \varepsilon_{n-3}\varepsilon_{n-2}\frac{k_{n-1}}{k_{n-2}}.$$

Hence it is shown that (16) is true for $i = 2$.

We now assume that Theorem (3.12) is truth for the case $i = p$. Then

$$H_{2p-2} = 0,$$

and

$$H_{2p-3} = \frac{k_{2m-2}}{k_{2m-3}} . \frac{k_{2m-4}}{k_{2m-5}} ... \frac{k_{2m+1-(2p-3)}}{k_{2m+1-(2p-2)}} \varepsilon_{2m-3}\varepsilon_{2m-4}...\varepsilon_{2m-(2p-2)}.$$

are satisfied. From Definition (3.2) we have

$$H_{2p-1} = \left(k_{2m-2p}H_{2p-3} - \nabla_{V_1} H_{2p-2}\right) \frac{\varepsilon_{2m-2p-2}\varepsilon_{2m-2p-1}}{k_{2m-2p-1}}.$$

By using of Definition (3.2) and hypothesis we have

$$H_{2p-1} = \frac{k_{2m-2}}{k_{2m-3}} . \frac{k_{2m-4}}{k_{2m-5}} ... \frac{k_{2m+1-(2p+1)}}{k_{2m+1-(2p+2)}} \varepsilon_{2m-3}\varepsilon_{2m-4}...\varepsilon_{2m-2p},$$

which completes the proof.

Definition 3.13. *Let M be a hypersurface in E_1^{2m}, and α be a non- degenerate curve on M. and $\{H_1, H_2, ..., H_{2m-3}\}$ be the harmonic curvature functions of the curve α. If the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \frac{k_6}{k_5} ... \frac{k_{2m-2}}{k_{2m-3}}$ are constant, then we have for $2 \leq i \leq m$ then the curve α is called LC slant helix in the sense of Hayden .*

So, we can give the following results:

Corollary 3.14. *Let M be a hypersurface in E_1^{2m}, and α be a non- degenerate curve on M and $\{H_1, H_2, ..., H_{2m-3}\}$ be the harmonic curvature functions of the curve α. If the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \frac{k_6}{k_5}, ..., \frac{k_{2m-2}}{k_{2m-3}}, \frac{k_{2m-2}}{k_{2m-3}}$ are constant, then from Theorem (3.7) and Theorem(3.12) we can easily see that the axis of a LC slant helix α is*

$$D = \varepsilon_0 H_{2m-3}V_1 + \varepsilon_2 H_{2m-5}V_3 + ... + \varepsilon_{2m-4}H_1 V_{2m-3} + \varepsilon_{2m-2}V_{2m-1}.$$

Corollary 3.15. *Let M be a hypersurface in E_1^{2m}, and α be a non-degenerate W-curve on M. By using Definition (3.13) and Corollary(3.14) α is a $V_{2m-1}-$LC slant helix in the sense of Hayden.*

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